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Optimal Transfers Between Coplanar Elliptical Orbits

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Optimal Transfer Equations

THE problem to be studied is that of optimal transfer of a rocket vehicle between a pair of coplanar elliptical orbits, employing two impulsive thrusts. The criterion for optimality is minimization of the propellant expenditure or, equivalently, of the characteristic velocity for the maneuver. If v_1 and v_2 denote the vector velocity increments generated in the vehicle by the impulsive thrusts, then the characteristic velocity V for the maneuver is defined by the equation

$$V = v_1 + v_2 \tag{1}$$

where v_1 and v_2 are the magnitudes of the increments. If c is the magnitude of the rocket jet velocity and R is the mass ratio for the maneuver, then it is well known that $c\ell nR = V$.

Taking the center of gravitational attraction as pole, the polar coordinates of the rocket will be denoted by $(1/u, \Theta)$, and it will be assumed that, for both terminal orbits, Θ increases with the time t. The polar equation of an orbit will be taken in the form

$$u = a + b\cos(\Theta - \omega) \tag{2}$$

where the constants a, b, and ω $(a > b > 0)$ will be termed the elements of the orbit. Denoting the semilatus rectum and the eccentricity of the orbit by ℓ and e, respectively, we have

$$\ell = 1/a, \qquad e = b/a \tag{3}$$

where ω is called the longitude of the periapse and is measured from a convenient reference line $\Theta = 0$.

The direction of an impulsive thrust I applied to a rocket situated at a point P $(1/u, \Theta)$ (Fig. 1) is determined by the angle ϕ it makes with the forward transverse direction (i.e., perpendicular to the radius vector OP). In the diagram, A is the periapse; ϕ is measured counterclockwise between 0 and 360 deg.

It has been proved[1] that, if the impulse transfers the rocket from an orbit (a, b, ω) into an orbit (a', b', ω'), then the characteristic velocity v for the transfer is given by

$$v = \gamma^{1/2} u (a'^{-1/2} - a^{-1/2}) \sec\phi \tag{4}$$

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where γu^2 is the gravitational acceleration toward the center of attraction O. Thus, if (a_1, b_1, ω_1) and (a_2, b_2, ω_2) are the elements of the terminal orbits, and (a, b, ω) are the elements of the transfer orbit, then the characteristic velocity V for the maneuver is given by

$$V = \gamma^{1/2} u_1 (a^{-1/2} - a_1^{-1/2}) \sec\phi_1 + \gamma^{1/2} u_2 (a_2^{-1/2} - a^{-1/2}) \sec\phi_2 \tag{5}$$

where $(1/u_1, \Theta_1)$ and $(1/u_2, \Theta_2)$ are the coordinates of the two junction points and (ϕ_1, ϕ_2) specify the directions of the impulsive thrusts.

If w denotes the component of rocket velocity in a direction perpendicular to the thrust (see Fig. 1), which is unaffected by the thrust, we define A by the equation

$$A = \gamma^{-1/2} w \operatorname{cosec}\phi \tag{6}$$

Then, if (A_1, A_2) are the respective values of A for the impulsive thrusts I_1, I_2, it has been shown[1,2] that for V to be stationary with respect to variations in the elements of the transfer orbit, it is necessary that

$$\left(\frac{u_1 + a}{A_1 a^{1/2}} + 1\right)\cos\phi_1 = \left(\frac{u_2 + a}{A_2 a^{1/2}} + 1\right)\cos\phi_2 \tag{7}$$

$$\left(\frac{u_1}{A_1} - A_1\right)\sin\phi_1 = \left(\frac{u_2}{A_2} - A_2\right)\sin\phi_2 \tag{8}$$

$$\begin{aligned}(u_1 - a)\left(1 + \frac{a^{1/2}}{A_1}\right)\cos\phi_1 + (u_1 - A_1 a^{1/2})\sin\phi_1\tan\phi_1 \\ = (u_2 - a)\left(1 + \frac{a^{1/2}}{A_2}\right)\cos\phi_2 + (u_2 - A_2 a^{1/2})\sin\phi_2\tan\phi_2\end{aligned} \tag{9}$$

In the same places, it was shown that A_1 and A_2 satisfy the equations

$$b_1\sin(\Theta_1 - \omega_1) = (u_1 - A_1 a_1^{1/2})\tan\phi_1 \tag{10}$$

$$b\sin(\Theta_1 - \omega) = (u_1 - A_1 a^{1/2})\tan\phi_1 \tag{11}$$

$$b_2\sin(\Theta_2 - \omega_2) = (u_2 - A_2 a_2^{1/2})\tan\phi_2 \tag{12}$$

$$b\sin(\Theta_2 - \omega) = (u_2 - A_2 a^{1/2})\tan\phi_2 \tag{13}$$

A further four equations are derived by substitution in the equations of the three orbits,

$$b_1\cos(\Theta_1 - \omega_1) = u_1 - a_1 \tag{14}$$

$$b\cos(\Theta_1 - \omega) = u_1 - a \tag{15}$$

$$b_2\cos(\Theta_2 - \omega_2) = u_2 - a_2 \tag{16}$$

$$b\cos(\Theta_2 - \omega) = u_2 - a \tag{17}$$

The last eleven equations are sufficient to determine sets of values for the 11 unknowns a, b, ω, u_1, u_2, Θ_1, Θ_2, A_1, A_2, ϕ_1, and ϕ_2 corresponding to stationary values of the characteristic velocity V [given by Eq. (5)]. Note that positive values of $a^{1/2}$, etc., are intended everywhere.

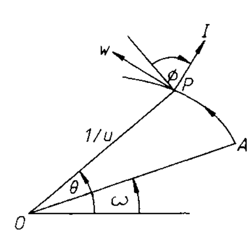

Fig. 1 Definition of thrust impulse angle ϕ.

Special Solutions of the Equations

Except for some special cases (e.g., terminal orbits whose axes are aligned), the 11 equations referred to in the previous section cannot be completely solved by algebraic manipulation, and recourse must be made to numerical methods. However, before turning to a consideration of such methods, we shall derive some special solutions that lead to stationary values for V but do not (in general) provide an absolute minimum value.

These are solutions for which $\phi_1 = -\phi_2 = \phi$, $u_1 = u_2 = u$, and $A_1 = A_2 = A$. It is immediately evident that the necessary conditions for V to be stationary [Eqs. (7-9)] are satisfied provided, in addition, we assume $u = A^2$. It remains to be shown that the remaining equations, Eqs. (10-17), can also be satisfied.

First, note that Eqs. (11), (12), (15), and (17) now reduce to

$$b \sin(\Theta_1 - \omega) = (A^2 - Aa^{1/2}) \tan\phi = -b \sin(\Theta_2 - \omega) \quad (18)$$

$$b \cos(\Theta_1 - \omega) = A^2 - a = b \cos(\Theta_2 - \omega) \quad (19)$$

Thus, either $b = 0$ or $\Theta_1 - \omega = -(\Theta_2 - \omega)$. If $b = 0$, then the transfer orbit is circular and, also, $A = a^{1/2}$; we reserve this case for later consideration. We assume, instead, that

$$\omega = ½(\Theta_1 + \Theta_2) \quad (20)$$

Equations (18) and (19) then require that

$$b \sin½(\Theta_1 - \Theta_2) = A(A - a^{1/2}) \tan\phi \quad (21)$$

$$b \cos½(\Theta_1 - \Theta_2) = A^2 - a \quad (22)$$

The remaining equations, Eqs. (10), (12), (14), and (16), take the forms

$$b_1 \sin(\Theta_1 - \omega_1) = A(A - a_1^{1/2}) \tan\phi \quad (23)$$

$$b_2 \sin(\Theta_2 - \omega_2) = -A(A - a_2^{1/2}) \tan\phi \quad (24)$$

$$b_1 \cos(\Theta_1 - \omega_1) = A^2 - a_1 \quad (25)$$

$$b_2 \cos(\Theta_2 - \omega_2) = A^2 - a_2 \quad (26)$$

Eliminating b between Eqs. (21) and (22), we find

$$a^{1/2} = A[\tan\phi \cot½(\Theta_1 - \Theta_2) - 1] \quad (27)$$

Dividing Eqs. (23) and (24), we get

$$\frac{b_1 \sin(\Theta_1 - \omega_1)}{b_2 \sin(\Theta_2 - \omega_2)} = \frac{A - a_1^{1/2}}{a_2^{1/2} - A} \quad (28)$$

Squaring this last equation and using Eqs. (25) and (26), we arrive finally at the equation

$$\frac{(A^2 - a_1)^2 - b_1^2}{(A^2 - a_2)^2 - b_2^2} = \left(\frac{A - a_1^{1/2}}{A - a_2^{1/2}}\right)^2 \quad (29)$$

This is a quintic equation for A that is readily solved for given values of a_1, b_1, a_2, and b_2 by a numerical procedure.

For each real root A, values for $(\Theta_1 - \omega_1)$ and $(\Theta_2 - \omega_2)$ follow from Eqs. (25) and (26). The signs of these two angles are ambiguous but are restricted by Eq. (28); two possibilities are always available. Since the magnitude of the cosine of a real angle cannot exceed unity, some roots for A may need to be discarded, thus reducing the number of transfers of this type.

Having calculated Θ_1 and Θ_2, $\tan\phi$ can be found from Eq. (23) or (24) and $a^{1/2}$ then follows from Eq. (27). Finally, b is determined by either of the equations (21) or (22). The circumstance that $a^{1/2}$ and b must both be positive further limits the possible solutions at this stage.

Substituting $u_1 = u_2 = u = A^2$, $\phi_1 = -\phi_2 = \phi$, into Eq. (5), we find that

$$V = \gamma^{1/2} A^2 (a_2^{-1/2} - a_1^{-1/2}) \sec\phi \quad (30)$$

where ϕ is still arbitrary to the extent of the addition of a multiple of π. This ambiguity is resolved by choosing ϕ to lie in the quadrant giving a positive value for V.

As an example, we shall take $a_1 = 3$, $b_1 = 1$, $\omega_1 = 0$, $a_2 = 2$, $b_2 = 1$, and $\omega_2 = 30$ deg. Then, to five places of decimals, the real roots for A are -1.60446, 1.57514, and 2.23882.

Taking $A = -1.60446$, we calculate that $\Theta_1 = 115.195$ deg, $\Theta_2 = -24.950$ deg or $\tan\phi = 0.16903$ and, in the second case, alternatively, $\Theta_1 = -115.195$ deg, $\Theta_2 = 84.950$ deg. In the first case, $\tan\phi = -0.16903$, and in the second case, $\tan\phi = 0.16903$. Then $a = 2.26846$ or 2.73120, and $b = 0.89732$ or 0.89714, respectively. The two corresponding values of ω follow from Eq. (20), viz., $\omega = 45.123$ deg or -15.123 deg. Thus, the elements of the transfer orbit are either (2.26846, 0.89732, 45.123 deg) or (2.73120, 0.89714, -15.123 deg).

Referring to Eq. (30), we now calculate that $V = 0.33877\gamma^{1/2}$ in both cases.

The remaining pair of roots for A are inadmissible. Thus, $A = 1.57514$ leads to negative values for $a^{1/2}$ and $A = 2.23882$ makes $\cos(\Theta_1 - \omega_1)$ and $\cos(\Theta_2 - \omega_2)$ both greater than 1.

Circular Transfer Orbits

As has been remarked, Eqs. (18) and (19) can be satisfied by taking $b = 0$ and, thereby, assuming the transfer orbit to be circular. These equations then further require that

$$A = a^{1/2} \quad (31)$$

Equations (10), (12), (14), and (16) then reduce as before to the forms of Eqs. (23-26), from which we derive Eqs. (28) and (29).

Having found the roots of Eq. (29) for A, $a^{1/2}$ follows from Eq. (31)—only positive roots for A are accordingly admissible. The remaining unknowns are then calculated as previously. Equation (20) is no longer valid, and ω remains indeterminate for the circular transfer orbit.

If, as before, we take $a_1 = 3$, $b_1 = 1$, $\omega_1 = 0$, $a_2 = 2$, $b_2 = 1$, and $\omega_2 = 30$ deg, then the real roots for A are again -1.60446, 1.57514, and 2.23882. Only the positive roots are relevant in this case.

If $A = 1.57514$, we calculate that $a = 2.48107$, $\Theta_1 = 121.261$ deg, $\Theta_2 = 91.245$ deg, and $\phi = -73.874$ deg. Alternative values are $\Theta_1 = -121.621$ deg, $\Theta_2 = -31.245$ deg, and $\phi = 73.874$ deg. In both cases $V = 1.15905\gamma^{1/2}$, a value that is clearly not optimal.

For the same reason as before, $A = 2.23882$ is inadmissible.

Terminal Orbits with Axes Aligned

In an earlier paper,[1] it was conjectured that, if the axes of the terminal orbits are aligned, then the axis of the optimal transfer orbit is also aligned with these axes; i.e., if $\omega_1 = 0$ or π, and $\omega_2 = 0$ or π, then $\omega = 0$ or π. In these circumstances, the two thrusts are applied at apses of the terminal orbits and the values of ϕ_1 and ϕ_2 are 0 or π, then $\omega = 0$ or π. It was further conjectured that, if other solutions of the optimizing equations exist, then these will result in larger values for the characteristic velocity.

We are now in a position to test these conjectures by calculating data for the special modes of transfer examined in the previous two sections for specific cases.

Thus, suppose $a_1 = 3$, $b_1 = 1$, $\omega_1 = 0$, $a_2 = 2$, $b_2 = 1$, and $\omega_2 = 0$. Then the roots of Eq. (29) for A have already been calculated to be -1.60446, 1.57514, and 2.23882, the last two being inadmissible for reasons set out earlier.

Taking $A = -1.60446$, we calculate for the noncircular transfer that

$$a = 2.49981, \qquad b = 0.86712, \qquad \omega = 30.123 \text{ deg}$$

$$\Theta_1 = 115.196 \text{ deg}, \qquad \Theta_2 = -54.951 \text{ deg}$$

$$\phi = 9.594 \text{ deg}, \qquad V = 0.33877\gamma^{1/2}$$

There are two transfer ellipses whose axes are aligned with those of the terminal orbits; these have elements (2.5, 1.5, 0) and (2.5, 0.5, 0). For all thrusts ϕ is zero, and Eq. (5) accordingly shows that the characteristic velocities for these two transfers are $0.29507\gamma^{1/2}$ and $0.33416\gamma^{1/2}$, respectively. Clearly, both are superior to the special mode of transfer, which is, however, a local minimum.

For the special circular transfer, we take $A = 1.57514$ and calculate

$$a = 2.48107, \qquad b = 0, \qquad \omega = ?$$

$$\Theta_1 = 121.261 \text{ deg}, \qquad \Theta_2 = 61.245 \text{ deg}$$

$$\phi = -73.874 \text{ deg}, \qquad V = 1.15904\gamma^{1/2}$$

resulting in a marked deterioration in V. Again, V is locally stationary but is not a minimum.

Numerical Calculation of Optimal Transfer

Although all of the transfers thus far considered are stationary relative to local variation of the elements of the transfer orbit, none is globally optimal. To derive the transfer leading to the smallest possible value for V, it is necessary to solve Eqs. (7-17) numerically in each specific case. An iterative procedure has been developed for this purpose.

Starting with trial values of a, Θ_1, and Θ_2, we calculate u_1 and u_2 from Eqs. (14) and (16). The values of b and ω then follow from Eqs. (15) and (17). Next A_1 and ϕ_1 are derived from Eqs. (10) and (11), and A_2 and ϕ_2 from Eqs. (12) and (13), similarly. The appropriate quadrants in which to locate ϕ_1 and ϕ_2 are determined by the requirement that the magnitudes of the velocity increments due to the two thrusts, calculated from Eq. (4), should be positive.

The differences between the left-hand and right-hand members of the stationarity conditions (7-9) can now be found. Denoting these differences by F, G, and H, the objective of an iterative procedure is to reduce them to zero. F, G, and H will all be functions of the trial values a, Θ_1, and Θ_2, and our intention is to increment these values by δa, $\delta\Theta_1$, and $\delta\Theta_2$, so that $F(a + \delta a,\ \Theta_1 + \delta\Theta_1,\ \Theta_2 + \delta\Theta_2) = G(a + \delta a,\ \Theta_1 + \delta\Theta_1,\ \Theta_2 + \delta\Theta_2) = H(a + \delta a,\ \Theta_1 + \delta\Theta_1,\ \Theta_2 + \delta\Theta_2) = 0$. We approximate these equations by working to the first order in the increments, to yield

$$\frac{\partial F}{\partial a}\delta a + \frac{\partial F}{\partial \Theta_1}\delta\Theta_1 + \frac{\partial F}{\partial \Theta_2}\delta\Theta_2 = -F(a, \Theta_1, \Theta_2)$$

$$\frac{\partial G}{\partial a}\delta a + \frac{\partial G}{\partial \Theta_1}\delta\Theta_1 + \frac{\partial G}{\partial \Theta_2}\delta\Theta_2 = -G(a, \Theta_1, \Theta_2)$$

$$\frac{\partial H}{\partial a}\delta a + \frac{\partial H}{\partial \Theta_1}\delta\Theta_1 + \frac{\partial H}{\partial \Theta_2}\delta\Theta_2 = -H(a, \Theta_1, \Theta_2) \tag{32}$$

Solving for δa, $\delta\Theta_1$, and $\delta\Theta_2$, new trial values of a, Θ_1, and Θ_2 are now calculated by application of the increments δa, etc., and the procedure is repeated. Provided the iteration converges, the increments generated by Eqs. (32) become more and more accurate and the quantities F, G, and H will tend to zero. The iteration is terminated immediately when the test function $\sqrt{(F^2 + G^2 + H^2)}$ falls below a prescribed value (in my own computer program, this value is taken to be 0.00001).

Before a computer program can be compiled, it is necessary to derive formulas for the partial derivatives $\partial F/\partial a$, etc. These are obtained by first partially differentiating all of the equations (10-17) with respect to a, Θ_1, and Θ_2, remembering that all of the quantities are being regarded as functions of these three variables. For example, differentiation of Eq. (11) with respect to Θ_1 yields

$$\frac{\partial b}{\partial \Theta_1}\sin(\Theta_1 - \omega) + b\cos(\Theta_1 - \omega)\left(1 - \frac{\partial \omega}{\partial \Theta_1}\right) = \left(\frac{\partial u_1}{\partial \Theta_1} - \frac{\partial A_1}{\partial \Theta_1}a^{1/2}\right)\tan\phi_1 + (u_1 - A_1 a^{1/2})\sec^2\phi_1 \frac{\partial \phi_1}{\partial \Theta_1} \tag{33}$$

Formulas for the partial derivatives of b, ω, u_1, u_2, A_1, A_2, ϕ_1, and ϕ_2 can easily be established from these equations. The calculation is laborious but is readily programmable on a computer.

Next, if

$$F = \left(\frac{u_1 + a}{A_1 a^{1/2}} + 1\right)\cos\phi_1 - \left(\frac{u_2 + a}{A_2 a^{1/2}} + 1\right)\cos\phi_2 \tag{34}$$

then F is seen to be expressed as a function of u_1, u_2, A_1, A_2, ϕ_1, ϕ_2, and a, each of which quantities is known to be a function of a, Θ_1, and Θ_2. Thus,

$$\frac{\partial F}{\partial a} = \frac{\partial F}{\partial u_1}\frac{\partial u_1}{\partial a} + \frac{\partial F}{\partial u_2}\frac{\partial u_2}{\partial a} + \frac{\partial F}{\partial A_1}\frac{\partial A_1}{\partial a} + \frac{\partial F}{\partial A_2}\frac{\partial A_2}{\partial a} + \frac{\partial F}{\partial \phi_1}\frac{\partial \phi_1}{\partial a} + \frac{\partial F}{\partial \phi_2}\frac{\partial \phi_2}{\partial a} + \frac{\partial F}{\partial a} \tag{35}$$

Here, in the left-hand member, F is to be regarded as a function of a, Θ_1, and Θ_2 alone, whereas in the right-hand member it is being treated as a function of u_1, u_2, A_1, A_2, ϕ_1, ϕ_2, and a, as indicated in Eq. (34). Equations for all of the partial derivatives present in Eq. (32) can be written down as at Eq. (35) and a computer program can be compiled to generate these quantities for any given trial values of a, Θ_1, and Θ_2. The iterative procedure is therefore fully programmable.

Having tested this procedure using various sets of values for the elements a_1, b_1, ω_1, a_2, b_2, and ω_2, it has been found that, if the initial trial values of a, Θ_1, and Θ_2 are too far away from a possible solution of our equations, the process may not converge. Furthermore, even if the process does converge, the rate of convergence is usually slow and the waiting time using a small home computer may be as much as 30 min. It is accordingly desirable to provide a preliminary computation that quickly generates a set of trial values approximating a possible solution.

To this end, we can treat the characteristic velocity V as given by Eq. (5) as a function of a, Θ_1, and Θ_2. Its partial derivatives $\partial V/\partial a$, $\partial V/\partial\Theta_1$, and $\partial V/\partial\Theta_2$ are immediately calculable, since formulas are available for the derivatives of u_1, u_2, ϕ_1, and ϕ_2. We then utilize an ordinary gradient minimization technique until V no longer decreases. The reigning values of a, Θ_1, and Θ_2 are then employed as trial values for the iterative procedure described earlier. Due to the very "flat" nature of the minima of the function $V(a, \Theta_1, \Theta_2)$, it is not possible to arrive at accurate optimal values of the variables by use of this gradient procedure.

It follows from the nature of this preliminary calculation that any set of trial values generated must lie in the vicinity of a minimum of the characteristic velocity. It can never suggest trial values corresponding to a nonminimal stationary value of V. Thus, applying it to the case considered earlier ($a_1 = 3$, $b_1 = 1$, $\omega_1 = 0$, $a_2 = 2$, $b_2 = 1$, $\omega_2 = 30$ deg), the circular transfer orbit solutions are never generated, indicating that these lead to nonminimal stationary values for V. If, however, a set of trial values in the vicinity of one of the other special solutions is offered, the process converges to this special solution; we conclude that these special transfers represent local minima of V.

By testing various sets of initial trial values with the gradient procedure and feeding from this to the iterative process, two further sets of solutions have been found to our equations in the case $a_1 = 3$, $b_1 = 1$, $\omega_1 = 0$, $a_2 = 2$, $b_2 = 1$, $\omega_2 = 30$ deg. The data for these are set out as follows:

$$a = 2.38929, \quad b = 1.37061, \quad \omega = 24.048 \text{ deg}$$

$$\Theta_1 = 61.245 \text{ deg}, \quad \Theta_2 = 185.085 \text{ deg}$$

$$u_1 = 3.48106, \quad u_2 = 1.09306$$

$$\phi_1 = 7.038 \text{ deg}, \quad \phi_2 = 8.425 \text{ deg}$$

$$A_1 = -2.09028, \quad A_2 = -1.23814$$

$$V = 0.31058\gamma^{1/2}$$

$$a = 2.51336, \quad b = 0.53909, \quad \omega = 12.244 \text{ deg}$$

$$\Theta_1 = 164.989 \text{ deg}, \quad \Theta_2 = 46.883 \text{ deg}$$

$$u_1 = 2.03413, \quad u_2 = 2.95690$$

$$\phi_1 = 3.257 \text{ deg}, \quad \phi_2 = 3.062 \text{ deg}$$

$$A_1 = -1.45305, \quad A_2 = -1.74807$$

$$V = 0.33488\gamma^{1/2}$$

Both solutions provide a local minimum for V, but the first is believed to represent the global optimal solution.

It appears, therefore, that in this case there are four minima for V and two stationary values (at least).

Conclusions

It has been shown that the equations determining the optimal two-impulse mode of transfer between coplanar elliptical orbits have many solutions, some of which yield values for the characteristic velocity of the maneuver that are local minima. These include special solutions that are readily derived from the equations and other solutions that can only be reached by use of an iterative computer program; the global optimal transfer is generally of the latter type. Solutions for which the transfer orbit is circular generate values of the characteristic velocity that are locally stationary but are not minima.

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Determining the Largest Hypersphere of Stability Using Lagrange Multipliers

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Introduction

ONE of the most important characteristics of a good control system is its ability to remain stable for uncertain but bounded parameter variations. This is referred to as stability robustness. Indeed, a primary reason for utilizing feedback control instead of open-loop control is the presence of model uncertainties. Uncertainties appear in the system mathematical model due to many factors, e.g., component tolerances, measurement errors, and linearization approximation. For example, if a mathematical model is developed for a system whose elements are known within a certain amount of tolerance, then the resulting system matrix can have entries that vary within some fixed bounds and around some nominal values. Thus, it is an important problem to obtain a quantitative scalar measure of allowable perturbations that can be tolerated by a system without affecting its stability.

Consider the state space model of a linear system:

$$\dot{x}(t) = (A^0 + E)x(t) = Ax(t) \tag{1}$$

where A^0 is the nominal system matrix and E is the perturbation matrix with a known structure. The perturbation as characterized by E is classified further into highly structured perturbation, in which the bounds on individual elements of the perturbation matrix are known, and weakly structured perturbation in which a bound on some norm of the perturbation matrix is known with no knowledge about the bounds of individual elements.

Robust stability criteria developed in the time domain are primarily based on the Lyapunov theory.[1-3] The results obtained carry some conservativeness for determining the upper bounds of the individual parameters as well as for finding the spectral norm bound of the perturbation matrix.[4]

The perturbation bound that we shall introduce next, namely, the radius of the largest hypersphere of stability, gives the designer the ability to obtain the maximum allowable Euclidean norm on the perturbation matrix elements under weakly structured perturbation as well as bounds on individual elements under highly structured perturbations. This will be accomplished by finding the boundary hypersurfaces that separate the parameter space into stable and unstable regions based on the results of Jury and Pavilidis.[5] By finding the shortest distance from the stable nominal point to the respective boundary hypersurfaces, the largest hypersphere centered at the nominal point is found in which elements of the system matrix can vary without affecting system stability. The effectiveness of the method will be illustrated by obtaining closed-form solutions for second-order systems and an example for a third-order system.

Main Results

Starting with the system description given in Eq. (1), the characteristic polynomial can be found by

$$f(s) = \det[sI - A] = b_n s^n + b_{n-1}s^{n-1} + \cdots b_1 s + b_0 \tag{2}$$

The coefficients b_i are functions of the elements of A. Based on Jury and Pavlidis' result,[5] the critical conditions for stability limits are given by

$$b_0 = 0 \tag{3a}$$

and

$$\Delta_{n-1} = \begin{bmatrix} b_{n-1} & b_{n-3} & b_{n-5} & b_{n-7} & \cdot & 0 \\ b_n & b_{n-2} & b_{n-4} & b_{n-6} & \cdot & \cdot \\ 0 & b_{n-1} & b_{n-3} & b_{n-5} & \cdot & \cdot \\ 0 & b_n & b_{n-2} & b_{n-4} & \cdot & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdot & b_0 \\ 0 & 0 & 0 & 0 & \cdot & b_1 \end{bmatrix} = 0$$

$$n > 1 \tag{3b}$$

where $\Delta_{n-1} \epsilon R^{(n-1)\times(n-1)}$. These two critical conditions for stability limits separate the parameter space into stable and

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